Math 105 Assignment 6 Solutions

1. Note each of the following problems, there are <u>many</u> way of doing them I am merely giving one solution.

a) $$\sum_{s=2015} \frac{4^{2s}\pi^{-s}}{e^s}$$
$$= \sum_{s=2015} \left(\frac{16}{\pi e}\right)^s$$

Since $\frac{16}{\pi e} > 1$, we have that the series is a geometric series with ratio bigger than 1. Thus the series diverges.

b) $$\sum_{a=5} \frac{1}{a \log^{99} a}$$

Since x, $\log^{99} x$ are positive increasing functions when $a \geq 5$, we have

$$\frac{1}{x \log^{99} x}$$

is a decreasing positive function, so by integral test, our series converges whenever the following integral does.

$$\int_5^\infty \frac{1}{x \log^{99} x}\,dx$$

Let's compute.

$$\int_5^{\infty} \frac{1}{x\log^{99}x}\,dx, \qquad u = \log x, \; du = \frac{1}{x}\,dx$$

$$x = 5 \Rightarrow u = \log 5$$
$$x \to \infty \Rightarrow u \to \infty$$

$$= \int_{\log 5}^{\infty} \frac{1}{u^{99}}\,du$$

$$= \left[-\frac{1}{98u^{98}} \right]_{\log 5}^{\infty}$$

$$= \lim_{b\to\infty} -\frac{1}{98u^{98}} + \frac{1}{98(\log 5)^{98}}$$

$$= \frac{1}{98\log^{98}5}$$

Thus our series converges.

c) We will show that our series is absolutely convergent.

$$0 \leq \left| \frac{\sin(2015\log i)}{i^{2015}} \right| \leq \frac{1}{i^{2015}}, \quad i \geq 4.$$

Since $\sum_{i=4}^{\infty} \frac{1}{i^{2015}}$ converges by p-test ($p = 2015$), we have by comparison test $\sum_{i=4}^{\infty} \left| \frac{\sin(2015\log i)}{i^{2015}} \right|$ converges.

Since absolute convergence implies convergence, we have

$$\sum_{i=4}^{\infty} \frac{\sin(2015\log i)}{i^{2015}}$$

converges.

d) $\displaystyle\sum_{f=1}^{\infty} \underbrace{\frac{(2f)!}{f^{2015}(f!)^2}}_{a_f}$

Let us apply ratio test.

$$\lim_{f\to\infty} \left| a_{f+1}/a_f \right| = \lim_{f\to\infty} \frac{(2(f+1))!}{(f+1)^{2015}(f+1)!^2} \Big/ \frac{(2f)!}{f^{2015}(f!)^2}$$

$$= \lim_{f\to\infty} \frac{(2f+2)!\ f!^2\ f^{2015}}{(2f)!\ (f+1)!^2\ (f+1)^{2015}}$$

$$= \lim_{f\to\infty} \frac{(2f+2)(2f+1)(2f)!\ f!^2}{(2f)!\left[(f+1)f!\right]^2} \left(\frac{f}{f+1}\right)^{2015}$$

$$= \lim_{f\to\infty} \frac{(2f+2)(2f+1)}{(f+1)^2} \left(\lim_{f\to\infty} \frac{f}{f+1}\right)^{2015}$$

Since highest order term in $\frac{f}{f+1}$ is f, we have

$$\frac{f}{f+1} = \frac{1}{1+\frac{1}{f}} \to 1 \text{ as } f\to\infty$$

Similarly the highest order term in

$$\frac{(2f+2)(2f+1)}{(f+1)^2}$$

is f^2 we divide top and bottom by the highest order terms.

$$\frac{(2f+2)(2f+1)}{(f+1)^2} = \frac{(2f+2)(2f+1)}{(f+1)^2}\frac{\frac{1}{f^2}}{\frac{1}{f^2}}$$

$$= \frac{\left(\frac{2f+2}{f}\right)\left(\frac{2f+1}{f}\right)}{\left(\frac{f+1}{f}\right)^2}$$

$$= \frac{\left(2+\frac{2}{f}\right)\left(2+\frac{1}{f}\right)}{\left(1+\frac{1}{f}\right)^2}$$

$$\to \frac{(2+0)(2+0)}{(1+0)^2} \quad \text{as } f\to\infty.$$

$$= 4$$

Thus $\lim_{f\to\infty}\left|\frac{a_{f+1}}{a_f}\right| = 4 > 1$ and thus the series diverges.

e) $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{n+4n\sqrt{n+5\sqrt{n}}}{\sqrt[3]{8n^4-en^6+\pi n^3-2015}}}_{a_n}$

We are already told that a_n is positive an decreasing when large.

Because of how ugly a_n is, we want to compare it to a b_n, and apply a limit comparison test. Let us find the asymptotic behaviour when n is large. When n is large, the highest order term dominate.

$$n+5\sqrt{n} \approx n$$

Thus $\quad n+4n\sqrt{n+5\sqrt{n}} \approx n+4n\sqrt{n}$

$$= n+4n^{3/2}$$

$$\approx 4n^{3/2}$$

Also when n is large,

$$8n^9 - en^6 + \pi n^3 - 2015 \approx 8n^9$$

thus.

$$\sqrt[3]{8n^3 - en^6 + \pi n^3 - 2015} \approx \sqrt[3]{8n^9} = 2n^3$$

So we have for large n,

$$a_n = \frac{n + 4n\sqrt{n + 5\sqrt{n}}}{\sqrt[3]{8n^9 - en^6 + \pi n^3 - 2015}} \approx \frac{4n^{3/2}}{2n^3} = \frac{2}{n^{3/2}}$$

Let $b_n = \frac{2}{n^{3/2}}$.

$$\frac{a_n}{b_n} = \frac{\dfrac{n + 4n\sqrt{n+5\sqrt{n}}}{\sqrt[3]{8n^9 - en^6 + \pi n^3 - 2015}}}{\dfrac{1}{n^{3/2}}}$$

$$= \frac{n^{5/2} + 4n^{5/2}\sqrt{n+5\sqrt{n}}}{\sqrt[3]{8n^9 - en^6 + \pi n^3 - 2015}}$$

$$= \frac{n^{5/2} + 4n^{5/2}\sqrt{n+5\sqrt{n}}}{\sqrt[3]{8n^9 - en^6 + \pi n^3 - 2015}} \quad \frac{\frac{1}{n^3}}{\frac{1}{n^3}}$$

$$= \frac{\dfrac{1}{\sqrt{n}} + 4\dfrac{\sqrt{n+5\sqrt{n}}}{\sqrt{n}}}{\sqrt[3]{\dfrac{8n^9 - en^6 + \pi n^3 - 2015}{n^9}}}$$

$$= \frac{\dfrac{1}{\sqrt{n}} + \sqrt{1 + \dfrac{5}{\sqrt{n}}}}{\sqrt[3]{8 - \dfrac{e}{n^3} + \dfrac{\pi}{n^6} - \dfrac{2015}{n^9}}}$$

So $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{0 + \sqrt{1+0}}{\sqrt[3]{8+0+0-0}}$

$= \dfrac{1}{2} \in (0, \infty)$

Thus $\sum\limits_{n=1}^{\infty} a_n$ converges whenever $\sum\limits_{n=1} b_n$ does.

Now $\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ converges by p-test ($p = 3/2$).

Thus $\sum\limits_{n=1}^{\infty} a_n$ converges by limit comparison test.

2. $\sum_{n=2}^{\infty} a_n$, is geometric implies $a_n = ar^n$ for some a, r.

a) $\left.\begin{array}{l} a_5 = 112 \\ a_7 = 7 \end{array}\right\} \Rightarrow \begin{array}{l} 112 = ar^5 \\ 7 = ar^7 \end{array}$

Dividing the 2 equations gives

$$r^2 = \frac{7}{112} = \frac{1}{16}$$

$$\Rightarrow r = \pm \frac{1}{4}$$

b) $r<0$ implies $r = -\frac{1}{4}$, so

$$a_2 = ar^2 = \frac{ar^5}{r^3} = \frac{a_5}{r^3} = \frac{112}{(-\frac{1}{4})^3} = -4^3 112 = -7168$$

$$a_9 = ar^9 = ar^7r^2 = a_7 r^2 = \frac{7}{16}$$

c) $\sum_{n=2}^{\infty} a_n = a_2 + a_3 + a_4 + \cdots$

$= a_2 + a_2 r + a_2 r^2 + \cdots$

$= a_2 (1 + r + r^2 + \cdots)$

$= a_2 \sum_{n=0}^{\infty} r^n$

$= \frac{a_2}{1-r}$ | since $|r|<1$

$= \frac{-7168}{1-(-\frac{1}{4})}$

$= \frac{-7168}{\frac{5}{4}}$

$= -\frac{28672}{5}$

Thus the series converges to $-\frac{28672}{5}$

3. a) $\sum_{n=3}^{\infty} \frac{n^2}{5^{n+1}} (x+10)^n$

Center is -10. To find radius of convergence, let's apply ratio test.

$$\left| \frac{(n+1)^2 (x+10)^{n+1}}{5^{(n+1)+1}} \Big/ \frac{n^2 (x+10)^n}{5^{n+1}} \right|$$

$$= \frac{(n+1)^2 \, |x+10|}{n^2 \; 5}$$

$$\to \frac{|x+10|}{5} \quad \text{as } n \to \infty$$

So $\frac{|x+10|}{5} < 1 \Rightarrow |x+10| < 5$

and the radius of convergence is 5.

b) $\sum_{n=2}^{\infty} \frac{n!}{n^2 n^n} (y-1)^n$

Center is 1. To find radius of convergence, apply ratio test.

$$\lim_{n\to\infty} \left| \frac{(n+1)! \, (y-1)^{n+1}}{(n+1)^2 (n+1)^{n+1}} \Big/ \frac{n!(y-1)^n}{n^2 n^n} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)! \; n^n \quad n^2 \quad |y-1|^{n+1}}{n! \, (n+1)^{n+1} (n+1)^2 \; |y-1|^n}$$

$$= \lim_{n\to\infty} (n+1) \frac{n^n}{(n+1)^{n+1}} \left(\frac{n}{n+1}\right)^2 |y-1|$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} \left(\frac{n}{n+1}\right)^2 |y-1|$$

$\lim_{n\to\infty} \frac{n}{n+1} = 1$, so we want to find the limit of

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \left[\left(\frac{n+1}{n}\right)^n\right]^{-1}$$

$$= \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^n\right]^{-1}$$

$$= e^{-1}$$

Thus the limit of the ratios is $\frac{|y-1|}{e}$.

$$\frac{|y-1|}{e} < 1 \Rightarrow |y-1| < e$$

So the radius of convergence is e.

c) $\displaystyle\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6 \cdots (2n-2)(2n)}{n!} x^n$

Clearly the center is 0. Again to find the radius of convergence, apply ratio test.

$$a_n = \frac{2\cdot 4 \cdots (2n-2)(2n)}{n!} x^n$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{2\cdot 4 \cdots (2n-2)(2n)(2n+2)\, x^{n+1}}{(n+1)!} \Big/ \frac{2\cdot 4 \cdots (2n-2)(2n)}{n!} x^n \right|$$

$$= \lim_{n\to\infty} \frac{2\cdot 4 \cdots (2n-2)(2n)(2n+2)}{2\cdot 4 \cdots (2n-2)(2n)} \frac{n!}{(n+1)!} |x|$$

$$= \lim_{n\to\infty} (2n+2) \frac{n!}{(n+1)!} |x|$$

$$= \lim_{n\to\infty} \frac{2n+2}{n+1} |x|$$

$$= 2|x|$$

$2|x| < 1 \Rightarrow |x| < \frac{1}{2}$

So the radius of convergence is $\frac{1}{2}$.

Alternatively you can notice that

$$2\cdot 4 \cdot 6 \cdots (2n-2)(2n) = (2\cdot 1)(2\cdot 2)(2\cdot 3) \cdots (2(n-1))(2n)$$
$$= 2^n (1\cdot 2\cdot 3 \cdots (n-1)(n))$$
$$= 2^n n!$$

Thus $a_n = \frac{2^n n!\, x^n}{n!} = 2^n x^n$

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (2x)^n$ which is geometric and converges when $|2x| < 1$

or $|x| < \frac{1}{2}$.

3. d) The center is 0.

To find the radius of convergence, let's apply ratio test

$$\lim_{n\to\infty} \left| \frac{f_{n+1} x^{n+1}}{f_n x^n} \right| = \lim_{n\to\infty} \frac{f_{n+1}}{f_n} |x| .$$

We can assume $\lim_{n\to\infty} \frac{f_{n+1}}{f_n}$ exits, so let $L = \lim_{n\to\infty} \frac{f_{n+1}}{f_n}$.

$$L = \lim_{n\to\infty} \frac{f_{n+1}}{f_n}$$

$$= \lim_{n\to\infty} \frac{f_n + f_{n-1}}{f_n}$$

$$= \lim_{n\to\infty} 1 + \frac{f_{n-1}}{f_n}$$

$$= \lim_{n\to\infty} 1 + \frac{1}{\left(\frac{f_n}{f_{n-1}}\right)}$$

$$= 1 + \frac{1}{\lim_{n\to\infty} \frac{f_n}{f_{n-1}}}$$

$$= 1 + \frac{1}{L}$$

Now we solve for L, since L satisfies

$$L = 1 + \frac{1}{L} \quad (A)$$

So $L^2 = L + 1$

$$\Rightarrow L^2 - L - 1 = 0$$

$$L = \frac{1 \pm \sqrt{1^2 - 4(1)(-1)}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

Since $f_{n+1} \geq f_n$ we have $\frac{f_{n+1}}{f_n} \geq 1$ and thus $L \geq 1$.

So $L = \frac{1 + \sqrt{5}}{2}$, since $\frac{1 - \sqrt{5}}{2} < 1$

Since the series converges when $L|x| < 1$ we have the radius of convergence is

$$|x| < \frac{1}{L} = L - 1 \quad (\text{by } \star)$$

$$= \frac{1 + \sqrt{5}}{2} - 1$$

$$= \frac{\sqrt{5} - 1}{2}$$

4. Since $\displaystyle\sum_{n=1}^{\infty} \frac{5a_{n+1}n^3 - 4\sqrt{n}}{a_n n^3 + 6n^2 - 2} = \sqrt[e]{\pi^{2015}}$

We have by the divergence test that

$$0 = \lim_{n\to\infty} \frac{5a_{n+1}n^3 - 4\sqrt{n}}{a_n n^3 + 6n^2 - 2}$$ highest order is n^3

$$= \lim_{n\to\infty} \frac{5a_{n+1}n^3 - 4\sqrt{n}}{a_n n^3 + 6n^2 - 2} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}}$$

$$= \lim_{n\to\infty} \frac{5a_{n+1} - \frac{4}{n^{5/2}}}{a_n + \frac{1}{n} - \frac{2}{n^3}}$$

$$= \lim_{n\to\infty} \frac{5a_{n+1} - 0}{a_n + 0 - 0}$$

$$= \lim_{n\to\infty} \frac{5a_{n+1}}{a_n}$$

Thus $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$, so by ratio test, $\displaystyle\sum_{n=1}^{\infty} a_n$ converges absolutely.

5. a) We have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/3)}{n!} (x-\pi/3)^n$$

$$= f(\pi/3) + f'(\pi/3)(x-\pi/3) + \frac{f''(\pi/3)}{2}(x-\pi/3)^2 + \frac{f'''(\pi/3)}{3!}(x-\pi/3)^3 + \cdots$$

$f(x) = \sin x$, so

$f(\pi/3) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$f'(\pi/3) = \cos\frac{\pi}{3} = \frac{1}{2}$

$f''(\frac{\pi}{3}) = -\sin\frac{\pi}{3} = \frac{-\sqrt{3}}{2}$

$f'''(\frac{\pi}{3}) = -\cos\frac{\pi}{3} = \frac{-1}{2}$

$f^{(4)}(\frac{\pi}{3}) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\vdots$

So $\sin(x) = \frac{\sqrt{3}}{2} + \frac{1}{2}(x-\frac{\pi}{3}) - \frac{\frac{\sqrt{3}}{2}}{2}(x-\frac{\pi}{3})^2 - \frac{\frac{1}{2}}{3!}(x-\frac{\pi}{3})^3 + \frac{\frac{\sqrt{3}}{2}}{4!}(x-\frac{\pi}{3})^4 + \frac{\frac{1}{2}}{5!}(x-\pi/3)^5 \cdots$

$$= \sum_{n=0}^{\infty} \frac{a_n}{n!}(x-\pi/3)^n, \quad a_n = \begin{cases} \frac{\sqrt{3}}{2} & , n = 4k \\ \frac{1}{2} & , n = 4k+1 \\ -\frac{\sqrt{3}}{2} & , n = 4k+2 \\ -\frac{1}{2} & , n = 4k+3 \end{cases}$$

To find the radius of we can do a ratio test or alternatively note.

$$|a_n| \leq 1$$

So $$\sum_{n=0}^{\infty} \left| \frac{a_n (x - \pi/3)^n}{n!} \right| \leq \sum_{n=0}^{\infty} \underbrace{\frac{|x - \pi/3|^n}{n!}}_{b_n}$$

$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n\to\infty} \left| \frac{|x - \pi/3|^{n+1}}{(n+1)!} \Big/ \frac{|x - \pi/3|^n}{n!} \right|$$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!} |x - \pi/3|^n$$

$$= \lim_{n\to\infty} \frac{|x - \frac{\pi}{3}|^n}{n+1}$$

$$= 0$$

So by ratio test the left hand side converges absolutely for all x.

Thus $\sum_{n=0}^{\infty} \left| \frac{a_n (x - \pi/3)^n}{n!} \right|$ converges by comparison test.

This implies that $\sum_{n=0}^{\infty} \frac{a_n (x - \pi/3)^n}{n!}$ converges absolutely for all x, and so the radius of convergence is ∞. So for all x,

$$\sin x = \sum_{n=0}^{\infty} \frac{a_n (x - \pi/3)^n}{n!}, \quad |x| < \infty$$

b) $x^{2015}e^{x^2}$, $x \geq 0$. Well we have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$$

By substituting x^2 for x we have

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}, \quad |x| < \infty$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}, \quad |x| < \infty$$

So $$x^{2015}e^{x^2} = x^{2015} \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}, \quad |x| < \infty$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+2015}}{n!}, \quad |x| < \infty.$$

$$= x^{2015} + x^{2017} + \frac{x^{2019}}{2} + \frac{x^{2021}}{3!} + \cdots$$

c) First let's find the power series of $\log(2015-x)$.

Since $$\left(\log(2015-x)\right)' = \frac{-1}{2015-x}$$

let's find the power series of $\frac{-1}{2015-x}$.

$$\frac{-1}{2015-x} = \frac{1}{2015} \frac{1}{1-\frac{x}{2015}}$$

$$= \frac{-1}{2015} \sum_{n=0}^{\infty} \left(\frac{x}{2015}\right)^n, \quad \left|\frac{x}{2015}\right| < 1$$

$$= \sum_{n=0}^{\infty} \frac{-x^n}{2015^{n+1}}, \quad |x| < 2015$$

$$\log(2015-x) = \int \frac{-1}{2015-x}\,dx$$

$$= \int \sum_{n=0}^{\infty} \frac{-x^n}{2015^{n+1}}\,dx \quad , |x| < 2015$$

$$= \sum_{n=0}^{\infty} \int \frac{-x^n}{2015^{n+1}}\,dx \quad , |x| < 2015$$

$$= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)2015^{n+1}} \quad , |x| < 2015$$

$$= C + \sum_{n=1}^{\infty} \frac{x^n}{n\,2015^n} \quad |x| < 2015$$

To find c, let $x=0$ so

$$\log(2015) = c$$

Thus $$\log(2015-x) = \log 2015 + \sum_{n=1}^{\infty} \frac{x^n}{n 2015^n} \quad , |x| < 2015$$

The power series of $x^2 \log(2015-x)$ is

$$x^2 \log(2015-x) = (\log 2015)\cdot x^2 + \sum_{n=1}^{\infty} \frac{x^{n+2}}{n\,2015^n} \quad , |x| < 2015$$

$$= (\log 2015)x^2 + \frac{x^3}{2015} + \frac{x^4}{2\cdot 2015^2} + \frac{x^5}{3\cdot 2015^3} + \cdots \quad , |x| < 2015$$

d). Let's first find the power series of

$$\frac{1}{2+x} \text{, centered at } x=1$$

$$\frac{1}{2+x} = \frac{1}{2-(1}$$

$$= \frac{1}{3+(x-1)}$$

$$= \frac{1}{3} \cdot \frac{1}{1-\left(-\frac{(x-1)}{3}\right)}$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} \left(-\frac{x-1}{3}\right)^n , \quad \left|-\frac{x-1}{3}\right| < 1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^n}{3^{n+1}} , \quad |x-1| < 3$$

Now let's take derivatives of both sides

$$\frac{-1}{(2+x)^2} = \frac{d}{dx}\left(\frac{1}{2+x}\right)$$

$$= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^n}{3^{n+1}} , \quad |x-1| < 3$$

$$= \sum_{n=0}^{\infty} \frac{d}{dx} \frac{(-1)^n (x-1)^n}{3^{n+1}} , \quad |x-1| < 3$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n n (x-1)^n}{3^{n+1}} , \quad |x-1 < 3$$

multiplying both sides by -2015 we get

$$\frac{2015}{(2+x)^2} = \sum_{n=0}^{\infty} \frac{2015(-1)^{n+1} n (x-1)^n}{3^{n+1}} , \ |x-1 < 3 = \frac{2015}{3} - \frac{2015 \cdot 2}{3^2}(x-1) + \frac{2015 \cdot 3}{3^3}(x-1)^2 \ldots$$